Journal of Statistical Physics, Vol. 98, Nos. 3/4, 2000

Critical Behavior for Maximal Flows on the Cubic Lattice

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Received March 30, 1999; final September 20, 1999

Let F_0 and F_m be the top and bottom faces of the box $[0, k] \times [0, l] \times [0, m]$ in Z^3. To each edge e in the box, we assign an i.i.d. nonnegative random variable $t(e)$ representing the flow capacity of e. Denote by $\Phi_{k,l,m}$ the maximal flow from F_0 to F_m in the box. Let p_c denote the critical value for bond percolation on Z^3. It is known that $\Phi_{k,l,m}$ is asymptotically proportional to the area of F_0 as $m, k, l \to \infty$, when the probability that $t(e) > 0$ exceeds p_c, but is of lower order if the probability is strictly less than p_c. Here we consider the critical case where the probability that $t(e) > 0$ is exactly equal to p_c, and prove that

$$\lim_{k,l,m \to \infty} \frac{1}{kl} \Phi_{k,l,m} = 0 \quad \text{a.s. and in } L_1$$

The limiting behavior of related to surfaces on Z^3 are also considered in this paper.

KEY WORDS: Maximal flows; critical behavior; surfaces.

1. INTRODUCTION

We begin with the notations in [K] to introduce flows through a medium. Consider the Z^3 lattice. To each edge $e \in Z^3$, we assign a random nonnegative value $t(e)$. It is assumed that all $t(e)$ are independent and have the same distribution function F. More formally, as a sample space we take

$$\Omega = \prod_{e \in Z^d} [0, \infty)$$

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and a product measure P on Ω. We interpret $t(e)$ as a capacity. In other words, $t(e)$ is the maximal amount of fluid which can flow through e per unit time. For given $t(e)$ we denote by $\Phi_{k,l,m}$ the maximal flow through the restriction of Z^3 to the box

$$B(k, l, m) := [0, k] \times [0, l] \times [0, m]$$

from its bottom face

$$F_0 := [0, k] \times [0, l] \times \{0\}$$

to its top face

$$F_m := [0, k] \times [0, l] \times \{m\}$$

Indeed, it follows from the definition in [K] that such a flow is an assignment of nonnegative numbers $g(e)$ and a direction to all the edges e in $B(k, l, m)$ such that

$$0 \leqslant g(e) \leqslant t(e) \qquad \text{for all } e$$

and such that for each vertex v outside $F_0 \cup F_m$ the total inflow equals the total outflow, that is,

$$\sum_v^+ g(e) = \sum_v^- g(e)$$

where $\sum^+$ ($\sum^-$) is the sum over all edges incident to v and directed towards v (away from v). For any such assignment, the flow from F_0 to F_m is defined as

$$\sum^+ g(e) - \sum^- g(e)$$

where $\sum^+$ ($\sum^-$) is the sum overall edges e with exactly one endpoint in F_m and e directed towards this endpoint (away from this endpoint). The maximum of this expression over all possible choices of $g(\cdot)$ is $\Phi_{k,l,m}$. The max-flow min-cut theorem allows us to express $\Phi_{k,l,m}$ in a different way. A set of edges E is said to separate F_0 from F_m in $B(k, l, m)$ if there is no path in $B(k, l, m) \backslash E$ from F_0 to F_m. We call E an (F_0, F_m) cut if E separates F_0 from F_m in $B(k, l, m)$ and if E is minimal in the sense that no proper subset of E separates F_0 from F_m. To each set of edges E we assign value

$$V(E) = \sum_{e \in E} t(e)$$

The max-flow and min-cut theorem (see [Bo]) states that

$$\Phi_{k,l,m} = \min\{V(E) : E \text{ an } (F_0, F_m) \text{ cut}\} \tag{1.1}$$

Before introducing some other functions, we need to introduce the *plaquettes* in the Z^3 lattice. For each edge e, we denote by $\pi(e)$ the unit square perpendicular to e and bisecting e. We shall call these unit squares of the form

$$[j_1 - \tfrac{1}{2}, j_1 + \tfrac{1}{2}] \times [j_2 - \tfrac{1}{2}, j_2 + \tfrac{1}{2}] \times [j_3 - \tfrac{1}{2}, j_3 + \tfrac{1}{2}] \quad \text{for} \quad j_i \in Z, \quad i = 1, 2, 3$$

Thus the plaquettes are square faces with the corners on

$$Z^3 + (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2})$$

We use $\mathscr{L}$ to denote these plaquettes. Clearly, plaquettes are associated in a one to one way to edges of Z^3. We set

$$t(e) = t(\pi)$$

if e is the edge associated to the plaquette π. Specially, if we have an edge set E on Z^3, we denote by E^* as the associated set in $\mathscr{L}$ and by ∂E^* the boundary of E^*. With these definitions, some other functions are also introduced by [K] in order to investigate the limit behavior of Φ. Let

$$\tau_{k,l} = \inf\{V(E) : E \text{ a cut over } [0, k] \times [0, l], \text{ where } \partial E^* \text{ consists of plaquettes of } \mathscr{L} \text{ on the perimeter of } [-\tfrac{1}{2}, k + \tfrac{1}{2}] \times [-\tfrac{1}{2}, l + \tfrac{1}{2}] \times \{\tfrac{1}{2}\} \text{ for } k, l \in Z\} \tag{1.2}$$

Here we say that a set E of edges of Z^3, or the set E^* of associated plaquettes, separates ∞ from $-\infty$ over S for a given set $S \subset Z^2$ if there is no path on Z^3 in $S \times Z \backslash E$ from $S \times \{-N\}$ to $S \times \{N\}$ for some (and hence all sufficiently large) $N > 0$. Similarly we call E or E^* a cut over S if E separates $-\infty$ from ∞ over S, but no proper subset of E separates $-\infty$ from ∞ over S. It follows from a subadditive argument (see [K]) that if

$$Ee^{rt(e)} < \infty \qquad \text{for some} \quad r > 0$$

then there exists a number $v = v(F) \leqslant Et(e)$ such that

$$\lim_{k,l \to \infty} \frac{1}{kl} \tau_{k,l} = v \qquad \text{a.s. and in } L_1 \tag{1.3}$$

Other functions are also introduced in [K].

$$\alpha_{k,l} = \inf\{V(E) : E \text{ separates } -\infty \text{ from } \infty \text{ over } [0,k]\times[0,l], \text{ where } \partial E^* \text{ consists of the plaquettes of } \mathscr{L} \text{ on the perimeter of } [-\tfrac{1}{2}, k+\tfrac{1}{2}]\times[-\tfrac{1}{2}, l+\tfrac{1}{2}]\times\mathbf{R} \text{ for } k, l \in Z\} \tag{1.4}$$

and

$$\beta_{k,l} = \inf\{V(E) : E \text{ separates } -\infty \text{ from } \infty \text{ over } [0,k]\times[0,l], \text{ where } \partial E^* \text{ contains the point} (-\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2})\} \tag{1.5}$$

Before we state the limit behaviors of Φ, α, and β, we need to introduce some basic percolation results. Consider percolation on the Z^3 lattice in which all edges are independently occupied with probability p and vacant with probability $1-p$. The value of an edge is also denoted by 1 or 0 if the edge is occupied or vacant. Next, we will define the occupied cluster. The cluster of vertex x, C_x, consists all vertices which are connected by an occupied path, where an occupied path is a nearest-neighbor path on Z^3 such that all of its edges are occupied. For brevity, we write C for the cluster of the origin. For any collection A of vertices, $|A|$ denoted the cardinality of A. The percolation probability is

$$\theta(p) = P_p(|C| = \infty)$$

and the critical probability is

$$p_c = \sup(p: \theta(p) = 0)$$

where P_p is the product measure on

$$\prod_{e \in Z^d} \{0, 1\}$$

Similarly, we can consider percolation on $\mathscr{H}$, where $\mathscr{H}$ is a graph whose vertices are in one-to-one correspondence with the plaquettes and two of whose vertices are adjacent if and only if the corresponding plaquettes intersect. We may think of these vertices as being located at the centers of the corresponding plaquettes, or at the midpoints of edges of Z^3. For example, the vertex at (0, 0, 1/2) has neighbors in $\mathscr{H}$ at $(\pm 1/2, 0, a)$, $(0, \pm 1/2, a)$, $(\pm 1, \pm 1/2, a)$, $(\pm 1/2, \pm 1, a)$ for $a = 0, 1$. By the translation invariance, we may choose a vertex, denoted by 0^* as the origin. Now we consider all vertices in $\mathscr{H}$ are independently occupied or vacant with probability p or $1-p$. The cluster of vertex x, $C^*(x)$ consists all vertices which are connected by an occupied path, where occupied path is a nearest-neighbor path on $\mathscr{H}$. Similarly, let

$$\theta^*(p) = P_p(|C^*(0^*)| = \infty)$$

and

$$p^* = \sup(p: \theta^*(p) = 0).$$

It is known that $1/27 \leqslant p^* \leqslant p_c$ (see [K]).

Let us return to discuss the flow problem. For any distribution F, we say e is vacant or occupied if $t(e) = 0$ or $t(e) > 0$. By comparing $\Phi_{k,l,m}$, $\alpha_{k,l}$ and $\beta_{k,l}$ with $\tau_{k,l}$, it is proved in [K] that if $F(0) < p^*$ and if $m(k,l) \to \infty$ as $k \geqslant l \to \infty$ in such a way that for some $\delta > 0$,

$$k^{-1+\delta} \log m(k,l) \to 0$$

then

$$\lim_{k,l,m\to\infty} \frac{1}{kl} \Phi_{k,l,m} = \lim_{k,l\to\infty} \frac{1}{kl} \tau_{k,l} = \lim_{k,l\to\infty} \frac{1}{kl} \alpha_{l,k}$$

$$= \lim_{k,l\to\infty} \frac{1}{kl} \beta_{k,l} = v \qquad \text{a.s. and in } L_1 \tag{1.6}$$

for $v > 0$. On the other hand, if $F(0) > 1 - p_c$, it is also proved in [K] that

$$\Phi_{k,l,m} = 0 \qquad \text{a.s. for all sufficiently large } k, l, m$$

and

$$\lim_{k,l\to\infty} \frac{1}{kl} \tau_{k,l} = \lim_{k,l\to\infty} \frac{1}{kl} \alpha_{l,k} = \lim_{k,l\to\infty} \frac{1}{kl} \beta_{l,k} = 0 \qquad \text{a.s. and in } L_1 \tag{1.7}$$

whenever $m(k,l) \to \infty$ as $k, l \to \infty$ in such a way that

$$\lim_{k,l\to\infty} \inf \frac{m(k,l)}{\log kl} > C$$

for some constant $C > 0$. It is conjectured that the limits in (1.6) also exist (see [K]) when $F(0)$ is between $[p^*, 1 - p_c]$. In fact, if we assume that limits in (1.6) exist in $[p^*, 1 - p_c)$, then it can be shown (see [CC]) that $v > 0$ or $v = 0$ if $F(0) < 1 - p_c$ or $F(0) > 1 - p_c$. In other words, $1 - p_c$ is a critical point for the maximal flow on $[0,k] \times [0,l] \times [0,m]$ since the flows range from nothing to $O(kl)$. Clearly, it is more interesting to ask the behavior of the flows at $1-p_c$. In fact, it is also conjectured by Harry Kesten (see [K]) that the limits in (1.6) exist and vanish at $1-p_c$. Here we answer these questions at $1-p_c$ affirmatively.

Theorem. If $F(0)=1-p_c$ and $F(0^-)=0$ and if in addition $Et(e)<\infty$, then for any $l>0$

$$\lim_{k,m\to\infty} \frac{1}{kl} \Phi_{k,l,m}=0 \tag{1.8}$$

for any $k>0$

$$\lim_{l,m\to\infty} \frac{1}{kl} \Phi_{k,l,m}=0 \tag{1.9}$$

and

$$\lim_{k,l,m\to\infty} \frac{1}{kl} \Phi_{k,l,m}=0 \tag{1.10}$$

where k, l, m in (1.10) go to ∞ without any restriction such as the condition in (1.6). Regarding to α and β,

$$\lim_{k,l\to\infty} \frac{1}{kl} \tau_{l,k}=\lim_{k,l\to\infty} \frac{1}{kl} \alpha_{l,k}=\lim_{k,l\to\infty} \frac{1}{kl} \beta_{l,k}=0 \qquad \text{a.s. and in } L_1 \tag{1.11}$$

Remarks. 1. The results in the theorem can be shown by using the same argument for any $d>3$.

2. The proof of the theorem cannot help us to show the existence of the limits in (1.6) when $F(0)\in[p^*, 1-p_c)$. In fact, it is possible to show the existence of the limits in (1.6) when $F(0)\in[0, p_c)$. But we do not have any idea when $F(0)\in(p_c, 1-p_c)$.

3. For $d=2$, most problems such as (1.6) and (1.8) are established in [GK].

Before giving a formal proof for the theorem, we would like to present an intuitive idea to show why the theorem is true. Let us consider a simpler case, that is $t(e)$ only takes value 1 or 0. In this case, $\Phi_{k,l,m}$ will be the number of disjoint occupied paths from F_0 to F_m inside $[0,k]\times[0,l]\times[0,m]$. For each such occupied path, if the starting vertex at F_0 is fixed, it follows from Theorem 1.1 in [BGN] that the probability of the existence of such a path goes to zero as $m\to\infty$. Then if there is some stationary property for the number of these paths, then the theorem will follows from a standard ergodic theorem. However, we do not have the stationary property for the number of the paths. Therefore, we have to use another stationary process $\sum_{x \in [0, k] \times [0, l]} R_x(m)$ (see the detailed definition in the following section) to control these paths. The theorem follows when we show

$$\lim_{k, l, m \to \infty} \frac{1}{kl} \sum_{x \in [0, k] \times [0, l]} R_x(m) = 0 \qquad \text{a.s. and in } L_1$$

2. PROOF

For any event A generated by $\{t(e): e \in S\}$, we denote by $T(A)$ the event by shifting each $e \in S$ one unit to the positive X direction. Clearly, T is measure preserving. On the other hand, it is easy to check

$$\lim_{n \to \infty} P(A \cap T^n(B)) = P(A)\, P(B)$$

for any events A and B, since we are working on the product space. Therefore, T is of mixing type. We will use this property later. Let $\theta_m^+(p)$ be the probability that there exists an infinite occupied cluster on

$$H = Z^2 \times \{0, 1,..., n,...\}$$

from the origin to the boundary of $[-m, m]^3 \cap H$. Then it follows from Theorem 1.1 in [BGN] that

$$\lim_{m \to \infty} \theta_m^+(p_c) = 0 \tag{2.1}$$

For each $x \in [0, \infty)^2 \times \{0\}$, let B_x be the five bonds in H which use x as their common vertex. Denote by

$$R_x(m) = \begin{cases} V(B_x) & \text{if there exists an occupied path from } x \text{ to } \{Z = m\} \text{ in } H \\ 0 & \text{otherwise} \end{cases}$$

where $\{Z = m\}$ is the plane with an equation that $Z = m$. Then $\{R_x(m)\}$ is stationary under the translation T. Clearly,

$$ER_x(m) \leqslant 5Et(e)\, \theta_m^+(p_c)$$

It follows from the Birkhoff ergodic theorem

$$\lim_{k \to \infty} \frac{1}{kl} \sum_{x \in [0, k] \times [0, l]} R_x(m) = ER_0 \leqslant 5\theta_m^+(p_c)\, E(e) \qquad \text{a.s. and in } L_1 \tag{2.2}$$

where 0 is the origin. Note that

$$R_x(m+1) \leqslant R_x(m) \tag{2.3}$$

For any $\varepsilon > 0$, we take m_0 such that

$$\theta^+_{m_0}(p_c) \leqslant \frac{\varepsilon}{6Et(e)}$$

After that, it follows from (2.2) that for m_0 we take k_0 such that

$$\frac{1}{kl} \sum_{x \in [0, k] \times [0, l]} R_x(m_0) \leqslant \varepsilon$$

for all $k \geqslant k_0$. It then follows from (2.3) that for all $m \geqslant m_0, k \geqslant k_0$

$$\frac{1}{kl} \sum_{x \in [0, k] \times [0, l]} R_x(m) \leqslant \frac{1}{kl} \sum_{x \in [0, k] \times [0, l]} R_x(m_0)$$

Therefore,

$$\lim_{k, m \to \infty} \frac{1}{kl} \sum_{x \in [0, k] \times [0, l]} R_x(m) = 0 \qquad \text{a.s. and in } L_1 \tag{2.4}$$

We also need to construct another variable R' as follows. For each $x \in [0, k] \times [0, l] \times \{0\}$,

$$R'_x(m) = \begin{cases} V(B_x) & \exists \text{ an occupied path from } x \text{ to } \{Z = m\} \\ & \qquad \text{in } [0, k] \times [0, l] \times [0, m] \\ 0 & \text{otherwise} \end{cases}$$

Clearly, R'_x may not be stationary but

$$R'_x(m) \leqslant R_x(m) \tag{2.5}$$

Now we would like to compare

$$\Phi_{k, l, m} \qquad \text{and} \qquad \sum_{x \in [0, k] \times [0, l]} R'_x(m)$$

Let I_x be the indicator of the event that there exists an occupied path from x to $\{Z = m\}$ in $[0, k] \times [0, l] \times [0, m]$ for $x \in [0, k] \times [0, l] \times \{0\}$. Let C^+_x be the occupied cluster in $[0, k] \times [0, l] \times [0, m]$ which contains x for $x \in [0, k] \times [0, l] \times \{0\}$ and let Δ_x be the outside vacant edge boundary of

C_x^+ in $[0,k]\times[0,l]\times[0,m]$. In other words, for any edge in Δ_x there exists an occupied path from one of its vertices to x in $[0,k]\times[0,l]\times[0,m]$ and there exists another path in $[0,k]\times[0,l]\times[0,m]$ from the other vertex to F_m without using any edge in C_x^+, and the edge is vacant. Denote by

$$Q=\bigcup_{\{x\in F_0\,:\,I_x=0\}}\Delta_x\bigcup_{\{x\in F_0\,:\,I_x\neq 0\}}B_x$$

Now we show that Q separates F_m from F_0 on $[0,k]\times[0,l]\times[0,m]$. Suppose that Q does not. Then there exists a path r inside $[0,k]\times[0,l]\times[0,m]$ from $x\in[0,k]\times[0,l]\times\{0\}$ to F_m which does not use any edge of Q. If $I_x=0$, we denote by

$$r_1=r\cap C_x^+\qquad\text{and}\qquad r_2=r\backslash r_1$$

Note that r_1 or r_2 may not be path anymore, but for each edge in r_1 there exists an occupied path from one of its vertices to x. On the other hand, since $I_x=0$, there does not exist an occupied path from x to F_m inside $[0,k]\times[0,l]\times[0,m]$ so that r_2 contains an edge such that the edge is connected to r_1 and there exists a path outside of C_x^+ from the edge to F_m. In fact, to see this, we can go along r from x to the last vertex in C_x^+ such that, after the vertex, the rest path of r is outside of C_x^+. The edge which connects the vertex in the rest path of r is what we are looking for. We write b for the edge in r_2. Note that if b is occupied, then it has to be in C_x. But by the definition of r_2, $b\notin C_x$ so that b cannot be occupied. On the other hand, there exist an occupied path connecting b to x inside C_x^+ and another path connecting b to F_m outside C_x^+ so that $b\in\Delta_x$. Therefore, since $\Delta_x\subset Q$, r has to use some edge in Q which contradicts the assumption. If $I_x\neq 0$, any path inside $[0,k]\times[0,l]\times[0,m]$ from x to F_m has to use at least one of bonds in B_x. Note that $B_x\subset Q$. Therefore, Q indeed separates F_0 from F_m. Let $Q_1\subset Q$ which is a minimal cut separating F_0 from F_m. Note that $V(\Delta_x)=0$ for each x so that

$$V(Q_1)\leqslant\sum_{x\in[0,k]\times[0,l]}R'_x(m)\tag{2.6}$$

On the other hand, Q_1 is a cut separating F_0 from F_m on $[0,k]\times[0,l]\times[0,m]$ so that

$$\Phi_{k,l,m}\leqslant V(Q_1)\tag{2.7}$$

Therefore, (1.8) in the theorem follows from (2.7), (2.5) and (2.4).

Clearly, by the symmetry and (1.8), (1.9) follows.

Now we show (1.10). It follows from the definition of $R_x(m)$ that

$$\sum_{x \in D_1 \cup D_2} R_x(m) = \sum_{x \in D_1} R_x(m) + \sum_{x \in D_2} R_x(m)$$

if $D_1 \cap D_2 = \varnothing$. It follows from a multi-parameter subadditive ergodic theorem (see Theorem 2.4 in [AK] and Theorem 1.1 in [S]) that

$$\lim_{k, l \to \infty} \frac{1}{kl} \sum_{x \in [0, k] \times [0, l]} R_x(m) = ER_0 \leqslant 5\theta_m^+(p_c)\, Et(e) \qquad \text{a.s. and in } L_1 \tag{2.8}$$

With this observation, (2.5)–(2.8), we have

$$\begin{aligned} 0 \leqslant \limsup_{k, l, m} \frac{1}{kl} \Phi_{k, l, m} &\leqslant \limsup_{k, l, m \to \infty} \frac{1}{kl} \sum_{x \in [0, k] \times [0, l]} R_x(m) \\ &\leqslant \lim_{k, l \to \infty} \frac{1}{kl} \sum_{x \in [0, k] \times [0, l]} R_x(m_0) \leqslant \varepsilon \end{aligned} \tag{2.9}$$

(1.10) follows from (2.9).

Now to show (1.11) we only need to show that

$$\lim_{k, l \to \infty} \frac{\tau_{k, l}}{kl} = 0 \qquad \text{a.s. and in } L_1$$

since $\tau_{k, l} \geqslant \alpha_{k, l}$ and $\tau_{k, l} \geqslant \beta_{k, l}$. We take $k \geqslant l_0$ and $l \geqslant l_0$ for l_0 that satisfies

$$\theta^+_{\log l_0}(p_c) \leqslant \varepsilon$$

for a given $\varepsilon > 0$. Let

$$R_1 = H \cap \{0\} \times [0, l] \times [0, \infty),$$

$$R_2 = H \cap \{k\} \times [0, l] \times [0, \infty),$$

$$R_3 = H \cap [0, k] \times \{0\} \times [0, \infty),$$

$$R_4 = H \cap [0, k] \times \{l\} \times [0, \infty)$$

that is the four side-faces of $[0, k] \times [0, l] \times [0, \infty)$ and let

$$R = R_1 \cup R_2 \cup R_3 \cap R_4$$

For each k, we divide $[0, k] \times [0, l]$ to

$$D_1 = [\log l, k - \log l] \times [\log l, l - \log l] \times \{0\}$$

and

$$D_2 = [0, k] \times [0, l] \times \{0\} \backslash D_1$$

Note that

$$D_1 \cup D_2 = F_0$$

For each $x \in D_1$, let

$$S_x = \begin{cases} V(B_x) & \exists \text{ an occupied path from } x \text{ to } R \text{ in } [0, k] \times [0, l] \times [0, \infty) \\ 0 & \text{otherwise} \end{cases}$$

For each $x \in D_2$, let $S_x = V(B_x)$. Note that $\{S_x\}$ is not stationary under T so that we need to construct another sequence. Let

$$S'_x = \begin{cases} V(B_x) & \exists \text{ an occupied path from } x \text{ to } x + B(\log l, \log l, \log l) \text{ in } H \\ 0 & \text{otherwise} \end{cases}$$

Then $\{S'_x\}$ is stationary under T on $[\log l, \infty) \times [\log l, l - \log l]$. Note that if there exists an occupied path from x to R in $[0, k] \times [0, l] \times [0, \infty)$ for $x \in D_1$, then there exists an occupied path from x to $x + B(\log l, \log l, \log l)$ so that

$$S_x \leqslant S'_x \qquad \text{for} \quad x \in D_1 \tag{2.10}$$

It follows from the Birkhoff ergodic theorem that

$$\lim_{k \to \infty} \frac{1}{(k - 2 \log l)(1 - 2 \log l)} \sum_{x \in D_1} S'_x$$
$$= ES'_0 \leqslant 5Et(e)\, \theta^+_{\log l}(p_c) \leqslant 5Et(e)\, \varepsilon \qquad \text{a.s. and in } L_1 \tag{2.11}$$

Clearly, by (2.10)

$$\lim_{k \to \infty} \frac{1}{k(l - 2 \log l)} \sum_{x \in D_1} S_x \leqslant 5Et(e)\, \theta^+_{\log l}(p_c) \leqslant 5Et(e)\, \varepsilon \qquad \text{a.s. and in } L_1 \tag{2.12}$$

Therefore, by (2.11)–(2.12)

$$\lim_{k \to \infty} \frac{1}{kl} \sum_{x \in F_0} S_x \leqslant 6Et(e)\,\varepsilon \qquad \text{a.s. and in } L_1 \tag{2.14}$$

Let I_x be the indicator of the event that there exists an occupied path from x to $\{R\}$ in $[0, k] \times [0, l] \times [0, \infty)$ for $x \in [0, k] \times [0, l] \times \{0\}$.

We now consider

$$Q = \bigcup_{\{x \in D_1 : I_x = 0\}} \Delta_x \bigcup_{\{x \in D_1, I_x \neq 0\}} B_x \bigcup_{x \in D_2} B_x \cap H$$

For each $u = (u_1, u_2, u_3)$ such that (u_1, u_2) is in the boundary of $[0, k] \times [0, l]$, then $u \in D_2$. Therefore, the edge which connects u and $v = (u_1, u_2, u_3 + 1)$ is in $B_u \subset Q$. With the observation

$$[-\tfrac{1}{2}, k + \tfrac{1}{2}] \times [-\tfrac{1}{2}, l + \tfrac{1}{2}] \times \{\tfrac{1}{2}\} \subset \partial Q^*$$

Now we show Q separates $-\infty$ from ∞ over $[0, k] \times [0, l]$. To show this, we suppose that Q does not. Then there exists a path, denoted by r from $x \in [0, k] \times [0, l] \times \{0\}$ to R without using Q. Suppose that $I_x \neq 0$. It would contradict since any path from x to R inside $[0, k] \times [0, l] \times [0, \infty)$ has to use one of bonds B_x. Now we suppose that $I_x = 0$. It follows from the same proof as we did for $\Phi_{k, l, m}$ that we could also get a contradiction. Therefore, Q separates $-\infty$ from ∞ over $[0, k] \times [0, l]$. Similarly, let $Q_1 \subset Q$ which is a minimum cut over $[0, k] \times [0, l]$ with

$$[-\tfrac{1}{2}, k + \tfrac{1}{2}] \times [-\tfrac{1}{2}, l + \tfrac{1}{2}] \times \{\tfrac{1}{2}\} \subset \partial Q_1^*$$

so that

$$\tau_{k, l} \leqslant V(Q_1) \leqslant V(Q) \tag{2.15}$$

Note that $V(\Delta_x) = 0$ so that

$$V(Q) = \sum_{x \in F_0} S_x$$

Therefore, it follows from (2.15) and (2.12) that for $l \geqslant l_0$

$$\lim_{k \to \infty} \frac{\tau_{k, l}}{kl} \leqslant 6Et(e)\,\varepsilon \qquad \text{a.s. and in } L_1 \tag{2.16}$$

We know (see (1.4) in [K]) that

$$\lim_{k, l \to \infty} \frac{\tau_{k, l}}{kl} = \inf_{k, l} \left\{ \frac{E\tau_{k, l}}{kl} \right\} \tag{2.17}$$

so that (1.11) follows from (2.16) and (2.17).

ACKNOWLEDGMENTS

The author would like to thank the referee for his numerous valuable comments. This research was supported by NSF Grant 9618128.

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